Forces in nonlinear media

2002 J. Phys. A: Math. Gen. 35 1437

(http://iopscience.iop.org/0305-4470/35/6/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 1437–1453 PII: S0305-4470(02)26284-8

Forces in nonlinear media

Mordehai Milgrom

Department of Condensed Matter Physics, Weizmann Institute, Rehovot, Israel

Received 27 June 2001, in final form 27 November 2001
Published 1 February 2002
Online at stacks.iop.org/JPhysA/35/1437

Abstract

I investigate the properties of forces on bodies in theories governed by the generalized Poisson equation $\vec{\nabla}\cdot[\mu(|\vec{\nabla}\varphi|/a_0)\vec{\nabla}\varphi] \propto G\rho$, for the potential φ produced by a distribution of sources ρ. This equation describes, *inter alia*, media with a response coefficient, μ, that depends on the field strength, such as in nonlinear, dielectric or diamagnetic, media; nonlinear transport problems with field-strength-dependent conductivity or diffusion coefficient; nonlinear electrostatics, as in the Born–Infeld theory; certain stationary potential flows in compressible fluids, in which case the forces act on sources or obstacles in the flow. The expressions for the force on a point charge are derived exactly for the limits of very low and very high charge. The force on an arbitrary body in an external field of asymptotically constant gradient, $-\boldsymbol{g}_0$, is shown to be $\boldsymbol{F} = Q\boldsymbol{g}_0$, where Q is the total effective charge of the body. The corollary $Q = 0 \Rightarrow \boldsymbol{F} = 0$ is a generalization of d'Alembert's paradox. I show that for $G > 0$ (as in Newtonian gravity) two point charges of the same (opposite) sign still attract (repel). The opposite is true for $G < 0$. I discuss its generalization to extended bodies and derive virial relations.

PACS numbers: 05.45.−a, 46.65.+g, 02.30.Jr, 03.50.De, 41.20.−q

1. Introduction

The Poisson equation, which governs so many physical processes, has the nonlinear generalization

$$\vec{\nabla}\cdot[\mu(|\vec{\nabla}\varphi|/a_0)\vec{\nabla}\varphi] = \alpha_D G\rho \tag{1.1}$$

by which the source distribution $\rho(\boldsymbol{r})$, in D-dimensional Euclidean space, gives rise to a potential field φ. Here, a_0 is a constant with the dimensions $\vec{\nabla}\varphi$, $\alpha_D = 2(\pi)^{D/2}/\Gamma(D/2)$, which is the D-dimensional complete solid angle, introduced here for convenience, and G is a coupling constant. As I will show, for $G > 0$, a point, test charge is attracted to a (finite) point charge of the same sign (as in gravity), whereas for $G < 0$ it is repelled.

Equation (1.1) describes a variety of physical problems; some examples are:

(i) Nonlinear dielectric, and diamagnetic, media; μ is then the dielectric or diamagnetic coefficient, which depends on the field strength (here $G < 0$).

(ii) Problems of nonlinear electric current flows in systems with field-dependent conductivity (nonlinear current–voltage relation), and nonlinear diffusion problems; $\mu(w)$ is the transport coefficient.

(iii) Stationary, subsonic, potential flow problems of non-viscous fluids with a barotropic equation of state $p = p(\varrho)$ (p is the pressure, ϱ the density). The stationary Euler equation is integrated into Bernoulli's equation $f(\varrho) = -\frac{1}{2}u^2 + \text{const}$, where $f'(\varrho) = \varrho^{-1}p'(\varrho) = c^2(\varrho)/\varrho$, with c the speed of sound; f thus increases with ϱ, and ϱ is a function of $|\boldsymbol{u}| = |\vec{\nabla}\varphi|$. The stationary continuity equation then gives $\vec{\nabla} \cdot [\varrho(|\vec{\nabla}\varphi|)\vec{\nabla}\varphi] = s(\boldsymbol{r})$, with s the source density (see e.g. [5] for the ideal-gas case). This is equation (1.1) with $\mu = \varrho$ and $G = \alpha_D^{-1} > 0$. For example, in a fluid with an equation of state of the form $p = a\varrho^\gamma$ ($a > 0, \gamma \geqslant 1$), we have $\varrho(u) = \varrho(0)\left[1 - (u/u_0)^2\right]^{1/(\gamma-1)}$, with $u_0^2 \equiv 2c_0^2/(\gamma - 1)$, and c_0 is the speed of sound at $u = 0$. Subsonicity requires $(u/u_0)^2 < (\gamma - 1)/(\gamma + 1)$.
If we, formally, consider a stationary flow problem in a medium with negative compressibility, $c^2 < 0$, ellipticity is maintained for any value of $\vec{\nabla}\varphi$. For example, for a medium with a constant $c^2 < 0$, $\varrho(u) = \varrho(0)\exp(u^2/2|c|^2)$.

(iv) Nonlinear (vacuum) electrostatics as formulated, e.g. in the Born–Infeld nonlinear electromagnetism, which also appears in effective Lagrangians resulting from string theory (see review and references in [1, 2]). In the original, electrostatic Born–Infeld theory $\mu(w) \propto (1 - w^2)^{-1/2}$, and $G < 0$.

(v) A formulation of an alternative nonrelativistic gravity to replace the dark-matter hypothesis in galactic systems [3]. Here $\mu(w) \approx w$ for $w \ll 1$, and $\mu \approx 1$ for $w \gg 1$ ($G > 0$).

(vi) Equation (1.1) was used in [4] as an effective-action approximation to Abelianized QCD.

(vii) Area (volume) minimization problems, such as the determination of the shape of a soap film with a dictated boundary (see, e.g., [5]): If $x_{D+1} = \varphi(x_1, \ldots, x_D)$ describes a D-dimensional hypersurface embedded in $(D + 1)$-dimensional Euclidean space with Cartesian coordinates $x_1, \ldots, x_{D+1}$, the volume element on the surface is $\mathrm{d}v = [1+(\vec{\nabla}\varphi)^2]^{1/2}\,\mathrm{d}^D r$. Then, equation (1.1) describes the problem of the minimization of the volume of the surface. The sources may describe a force density on the hypersurface acting in the direction x_{D+1}. In this problem $\mu(w) \propto (1 + w^2)^{-1/2}$, and $G > 0$. Born–Infeld electrostatics is the same as the area-extremization problem for a time surface embedded in Minkowski space-time.

Much has been said in the mathematical literature on the properties of the potentials that solve equation (1.1) (see, e.g., [5]). However, to my knowledge, very little has been said about forces on bodies in such theories. The forces can be written as certain integrals of φ, and are of obvious relevance in the physics context.

Many of the familiar and intuitive properties of the linear theory are lost in the nonlinear case because the potential, and forces, are not the sum of the contributions of the subsystems. For example, the force on a point charge is no longer proportional to the charge, it does not reverse direction when the charge of the body reverses sign, etc. And Earnshaw's theorem [6] no longer holds. Under some circumstances it is possible to suspend stably static charged bodies in a static field. This last aspect is treated in detail in [7].

The choice $\mu(w) = w^{D-2}$ is special in that the theory is then conformally invariant, and lends itself to many analytical developments. This case is described in detail in [8].

After discussing some general aspects of equation (1.1) in section 2, I take up the main subject concerning the properties of forces on bodies in nonlinear theories: general properties in sections 3, and forces on point charges in section 4. I conclude in section 5 with some examples of applications.

2. General properties

2.1. Preliminaries

The field equation (1.1) is derivable from the action functional

$$S = S_i + S_f \equiv -\int_V \rho\varphi \, \mathrm{d}^D r - \frac{a_0^2}{2\alpha_D G} \int_V \mathcal{F}\left[(\vec{\nabla}\varphi)^2/a_0^2\right] \mathrm{d}^D r. \tag{2.1}$$

The function $\mu(w)$ in equation (1.1) is given by

$$\mu(w) = \frac{\mathrm{d}\mathcal{F}(y)}{\mathrm{d}y} \qquad y = w^2. \tag{2.2}$$

For equilibrium problems, such as (i), (iv), (v) and (vii) above, the quantity $E \equiv -S$ may be identified with the energy of a charge configuration ρ when S is finite. I shall only be interested in differences in the energy between configurations with the same total charge, and this is finite, in general, even if the expression for E diverges. In the case of non-equilibrium, stationary transport problems, such as nonlinear diffusion, heat transfer, etc, E is not an energy but is related to the entropy-generation rate.

For convenience, the free, additive constant in $\mathcal{F}$ is chosen such that $\mathcal{F}(0) = 0$. This makes the field-action density vanish at $\boldsymbol{r} \to \infty$ when $\vec{\nabla}\varphi \to 0$, and improves convergence. I exclude from discussion theories for which $\mathcal{F}$ diverges at zero argument. I also assume $\mu(w) > 0$ except, possibly, at $w = 0$ where μ may vanish. Thus $\mathcal{F}(y)$ is an increasing function and is positive for $y > 0$. (I assume that $\mathcal{F}'(y)$ does not vanish for $y > 0$; it then has a uniform sign which can always be taken as positive by adjusting the sign of G.)

In the area-minimization case $\mathcal{F}(y) = (1 + y)^{1/2} - 1$. In the flow problem $\mathcal{F}(y)$ is essentially the equation of state since the pressure is given by $p = p(u = 0) - \frac{u_0^2}{2}\mathcal{F}\big[u^2(\varrho)/u_0^2\big]$, with the additive constant chosen so that $\mathcal{F}(0) = 0$ $[p \leqslant p(0)]$.

All that I say below is generalized in a straightforward manner to the case where $\mathcal{F}$, and thus μ, depend explicitly on $\boldsymbol{r}$. For instance, in the flow problem this happens when the fluid is coupled to some potential field ψ via $\varrho(\boldsymbol{r})\psi(\boldsymbol{r})$ for which the Bernoulli equation becomes $f(\varrho) = -\frac{1}{2}u^2 - \psi(\boldsymbol{r}) + \text{const.}$. To avoid encumbrance I assume all along, unless otherwise stated, that there is no such explicit $\boldsymbol{r}$ dependence.

When the charges are not held fixed but move under the influence of the φ field, and their dynamics is of interest, we add to the action the kinetic term for the charges

$$S_p = \frac{1}{2}\int \mathrm{d}^D r \varrho_m v^2(\boldsymbol{r}) \tag{2.3}$$

where ϱ_m is the mass density of the particles contributing ρ to the charge density. Extremizing $S + S_p$ with respect to particle coordinates gives the usual Euler equation $\varrho_m \dot{\boldsymbol{v}} = -\rho\vec{\nabla}\varphi$.

The solution of equation (1.1) is unique inside a volume V when one dictates on its boundary the value of φ or that of $\mu(|\vec{\nabla}\varphi|/a_0)\partial_n\varphi$ (or a combination thereof) ($\partial_n\varphi$ is the normal component of φ) provided the function $\nu(w) \equiv w\mu(w)$ is an increasing function (see e.g. [9] for a proof). This monotonicity of $\nu(w)$—which I shall assume all along—is tantamount to $\hat{\mu} \equiv \mathrm{d}\ln\mu(w)/\mathrm{d}\ln w > -1$. This is also the condition for the ellipticity of the field equation (2.11). In the case of a stationary flow we have $\hat{\mu} = -u^2/c^2$; the ellipticity condition is then equivalent to the subsonicity of the flow.

I now show that from the ellipticity condition and the choice $\mathcal{F}(0) = 0$ follows that the logarithmic derivative of $\mathcal{F}$, $\hat{\mathcal{F}}(y) \equiv y\mathcal{F}'(y)/\mathcal{F}(y)$, satisfies $\hat{\mathcal{F}}(y) > 1/2$ for all $y > 0$ for which ellipticity obtains. I use this inequality repeatedly in what follows. Define

$$\chi(w) \equiv \mathcal{F}(w^2)[2\hat{\mathcal{F}}(w^2) - 1] = 2\mu(w)w^2 - \mathcal{F}(w^2) \tag{2.4}$$

(where $y = w^2$). First note that $\chi'(w) = 2w\mu(1 + \hat{\mu})$, so $\chi' > 0$ for $w > 0$. Since $\chi(0) = 0$, $\chi(w) > 0$ for $w > 0$. Thus $\hat{\mathcal{F}}(y) > 1/2$ for all $y > 0$ [as $\mathcal{F}(y) > 0$ for $y > 0$]. (The condition $\mathcal{F}(0) = 0$, on which the derivation of $\hat{\mathcal{F}} > 1/2$ depends, does indeed enter in the cases where this inequality is used below.)

It is useful to write our theory for a general curved space whose metric is g_{ij} (g^{ij} its inverse, and $g = |\det(g_{ij})|$). The covariant form of the action is then

$$S = -\int_V g^{1/2}\rho^*\varphi \, \mathrm{d}^D r - \frac{a_0^2}{2\alpha_D G}\int_V g^{1/2}\mathcal{F}\left[\left(\varphi_{,}{}^{i}\varphi_{,i}\right)/a_0^2\right] \mathrm{d}^D r \tag{2.5}$$

with $\varphi_{,}{}^{i} = g^{ij}\varphi_{,j}$ ($\varphi_{,i} \equiv \partial\varphi/\partial x^i$), and $\rho^* \equiv g^{-1/2}\rho$ is a scalar under general coordinate transformations. Repeated indices are summed over. The covariant form of the field equation is

$$\left[\mu\left(\varphi_{,}{}^{i}\varphi_{,i}/a_0^2\right)\varphi_{,}{}^{k}\right]_{;k} = \alpha_D G\rho^* \tag{2.6}$$

where a semicolon signifies a covariant derivative. The covariant divergence appearing in equation (2.6) is given in terms of the normal divergence of a vector v^k as $v^k{}_{;k} = g^{-1/2}(g^{1/2}v^k)_{,k}$. So, using usual derivatives instead we have

$$\left[g^{1/2}\mu\left(\varphi_{,}{}^{i}\varphi_{,i}/a_0^2\right)\varphi_{,}{}^{k}\right]_{,k} = \alpha_D G\rho. \tag{2.7}$$

From the covariant action we can derive the field stress tensor (the energy–momentum tensor when working in Lorentzian space–time). This is the functional derivative of the field action with respect to the metric: under a variation δg_{ij}

$$\delta S_f = \frac{1}{2}\int g^{1/2}\delta g^{ij}\mathcal{P}_{ij} \, \mathrm{d}^D r. \tag{2.8}$$

In the Euclidean case, on which I concentrate hereafter, we find

$$\overset{\leftrightarrow}{\mathcal{P}} = -\frac{a_0^2}{2\alpha_D G}(\mathcal{F} - 2\mu\vec{\nabla}\varphi \otimes \vec{\nabla}\varphi) = -\frac{a_0^2}{2\alpha_D G}\mathcal{F}(1 - 2\hat{\mathcal{F}}\boldsymbol{e}\otimes\boldsymbol{e}) \tag{2.9}$$

where $\boldsymbol{e} \equiv \vec{\nabla}\varphi/|\vec{\nabla}\varphi|$ is a unit vector along $\vec{\nabla}\varphi$. The trace of $\overset{\leftrightarrow}{\mathcal{P}}$ is $-\left(a_0^2/2\alpha_D G\right)\mathcal{F}(D - 2\hat{\mathcal{F}})$. The field direction $\boldsymbol{e}$ is an eigenvector of $\overset{\leftrightarrow}{\mathcal{P}}$ with eigenvalue $-\left(a_0^2/2\alpha_D G\right)\mathcal{F}(1 - 2\hat{\mathcal{F}})$. The inequality $\hat{\mathcal{F}} > 1/2$ implies that this eigenvalue is always positive: there is always tension along the field lines. All other eigenvalues are equal and negative.

For solutions of the field equation the divergence of $\mathcal{P}_{ij}$, which measures the rate of change of the momentum density, is given by

$$\vec{\nabla}\cdot\overset{\leftrightarrow}{\mathcal{P}} = \rho\vec{\nabla}\varphi. \tag{2.10}$$

This conservation law can be derived directly from the Euclidean action (2.1) and follows from its translation invariance: under infinitesimal translations $\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{a}$, $\varphi(\boldsymbol{r}) \to \varphi(\boldsymbol{r}) + (\boldsymbol{a}\cdot\vec{\nabla})\varphi$, etc $S \to S + \boldsymbol{a}\cdot(\vec{\nabla}\cdot\overset{\leftrightarrow}{\mathcal{P}} - \rho\vec{\nabla}\varphi)$.

The field equation can also be written as

$$\mu\mathcal{A}_{ij}\varphi_{,i,j} = \alpha_D G\rho \tag{2.11}$$

where

$$\overset{\leftrightarrow}{\mathcal{A}} = (1 + \hat{\mu}\boldsymbol{e}\otimes\boldsymbol{e}) \tag{2.12}$$

with $\hat{\mu}$ the logarithmic derivative of μ, and $\boldsymbol{e}\otimes\boldsymbol{e}$ is the matrix whose (i, j) element is $e_i e_j$ (all dependent on $\vec{\nabla}\varphi$). Since $\hat{\mu} > -1$ $\overleftrightarrow{\mathcal{A}}$ is positive definite. If $\mu(0) = 0$, points where $\vec{\nabla}\varphi = 0$ need special treatment which I do not go into here.

If we make a small change $\delta\rho$ in ρ, the field equation can be linearized in the small change ζ in φ to read [9]

$$\vec{\nabla}\cdot[\mu\overleftrightarrow{\mathcal{A}}\cdot\vec{\nabla}\zeta] = \alpha_D G\delta\rho \tag{2.13}$$

This is the same as the equation for the electrostatic potential produced by the density $\delta\rho$ in a linear dielectric medium with a position-dependent, anisotropic dielectric constant $\mu\overleftrightarrow{\mathcal{A}}$.

A variation of φ in the volume V gives rise to a variation in S

$$\delta S = \frac{1}{\alpha_D G}\int_V \delta\varphi\{\vec{\nabla}\cdot[\mu(|\vec{\nabla}\varphi|/a_0)\vec{\nabla}\varphi] - \alpha_D G\rho\}\,\mathrm{d}^D r - \frac{1}{\alpha_D G}\int_\Sigma \mu\delta\varphi\vec{\nabla}\varphi\cdot\mathrm{d}\boldsymbol{s} \tag{2.14}$$

where Σ is the boundary of V. For potentials that solve the field equation, and hence nullify the first term in equation (2.14), we have

$$\delta S = -\frac{1}{\alpha_D G}\int_\Sigma \mu\,\delta\varphi\vec{\nabla}\varphi\cdot\mathrm{d}\boldsymbol{s}. \tag{2.15}$$

We can obtain useful integral constraints on such solutions—such as conservation laws and virial relations—by considering specific variations that do not nullify the surface term. Some examples are given in appendix A.

The second-order change in the action (energy) is

$$\delta^2 E = -\delta^2 S = \frac{1}{2\alpha_D G}\int_V \mu\vec{\nabla}\delta\varphi\cdot\overleftrightarrow{\mathcal{A}}\cdot\vec{\nabla}\delta\varphi\,\mathrm{d}^D r. \tag{2.16}$$

The ellipticity condition makes the integral positive (when $\vec{\nabla}\varphi \not\equiv 0$), and thus a solution of the field equation is a minimum of the energy for $G > 0$, and a maximum for $G < 0$.

Using the integral relation (A.1) derived in appendix A, which holds for solutions of the field equation, to eliminate the explicit dependence of E on the sources we get

$$E = -S = -\frac{a_0^2}{2\alpha_D G}\int_V \mathcal{F}\left[(\vec{\nabla}\varphi)^2/a_0^2\right](2\hat{\mathcal{F}} - 1)\,\mathrm{d}^D r. \tag{2.17}$$

So, in light of the above inequality for $\hat{\mathcal{F}}$, E is positive for $G < 0$ (as in electrostatics).

The field equation enjoys a certain scaling property [9] in that if $\varphi(\boldsymbol{r})$ and $\rho(\boldsymbol{r})$ are a consistent pair then so are

$$\varphi_\lambda(\boldsymbol{r}) = \lambda\varphi(\lambda^{-1}\boldsymbol{r}) \qquad \rho_\lambda(\boldsymbol{r}) = \lambda^{-1}\rho(\lambda^{-1}\boldsymbol{r}) \tag{2.18}$$

with appropriately scaled boundary conditions. Charges then scale as $q_\lambda = \lambda^{D-1}q$.

2.2. Asymptotic behaviour of the potential

When the medium can be considered infinite, a common choice of boundary condition, describing an isolated system, is $\vec{\nabla}\varphi \to 0$ at infinity. In fact, if the potential can be assumed to become spherical at infinity, this boundary requirement sometimes follows from the potential equation itself through Gauss's theorem. The behaviour of $\mu(w)$ near $w = 0$ is then relevant. For concreteness, I assume in the rest of the paper that $\mathcal{F}$, and thus μ, approach a power of the argument near 0.

$$\mu(w) \to w^{\beta_0} \qquad \mathcal{F}(y) \to \frac{2}{2+\beta_0}y^{(2+\beta_0)/2} \qquad (\beta_0 > -1). \tag{2.19}$$

If the sources ρ are contained within a finite volume, and the total charge, Q, does not vanish, the field becomes radial at infinity, and, applying Gauss's theorem to the field equation for a sphere of a large radius r, we find asymptotically

$$\vec{\nabla}\varphi \approx s(QG)|\hat{G}|^{\gamma_0}|Q|^{\gamma_0} r^{-\gamma_0(D-1)}\boldsymbol{n}. \tag{2.20}$$

Here $\hat{G} \equiv G a_0^{\beta_0}$, $\gamma_0 \equiv 1/(1+\beta_0)$, $s(a) = \mathrm{sign}(a)$, and $\boldsymbol{n} \equiv \boldsymbol{r}/|\boldsymbol{r}|$.

The value $\beta_0 = D-2$ is limiting. For higher values of β_0, φ diverges like a power of r at large r (when $Q \neq 0$) with $\vec{\nabla}\varphi$ still vanishing there. For the limiting case, φ is logarithmic at infinity and $\vec{\nabla}\varphi \propto \boldsymbol{r}r^{-2}$. In this case the action is infinite when calculated for the whole space. The revised theory of Newtonian gravity discussed in [3] is of this limiting-power type, and so is the class of conformally invariant theories with $\hat{\mu} = D-2$ discussed in [8]. In what follows I assume $\beta_0 \leqslant D-2$ unless otherwise stated (as in the discussion of one-dimensional systems).

For $Q = 0$ I do not have a general expression for the asymptotic behaviour of the field. Because of the nonlinearity, multipoles are no longer very relevant. For $\beta_0 = D-2$ the asymptotic form can be obtained as follows. In the exact-power-law theory with $\beta = D-2$ one can use the conformal invariance. Take the origin at a point outside charges and make a conformal transformation $\boldsymbol{r} \to a^2\boldsymbol{r}/r^2$, where a is the radius of the reflection sphere. If $\varphi(\boldsymbol{r})$ is the solution of the original problem then $\hat{\varphi}(\boldsymbol{r}) \equiv \varphi(a^2\boldsymbol{r}/r^2)$ is the solution of the new problem with the transformed charge distribution $\hat{\rho}(\boldsymbol{r}) = (a/r)^{2D}\rho(a^2\boldsymbol{r}/r^2)$, which also has a vanishing total charge (see [8] for more details). So the asymptotic behaviour of φ is obtained from the behaviour of $\hat{\varphi}$ at the origin (where the charge density is 0). I assume that $\hat{\varphi}$ is analytic there, so its dominant behaviour is, generically, $\hat{\varphi} \approx a^{-2}\boldsymbol{K}\cdot\boldsymbol{r}$, where $\boldsymbol{K}$ is some constant vector. This provides the generic, asymptotic behaviour of φ

$$\varphi \approx \boldsymbol{K}\cdot\boldsymbol{r}/r^2. \tag{2.21}$$

$\boldsymbol{K}$ might be viewed as the asymptotic-behaviour dipole, but it is not proportional to the dipole of the charge distribution. If $\boldsymbol{K} = 0$, the asymptotic behaviour is, more generally, of the form

$$\varphi \approx K_{i_1\cdots i_n} r_{i_1}\cdots r_{i_n}\big/r^{2n} \tag{2.22}$$

where $a^{-2n}K_{i_1\cdots i_n}$, a totally symmetric constant tensor, is the first non-vanishing Taylor coefficient in the expansion of $\hat{\varphi}$ at the origin. (I am discussing only the leading behaviour, not the expansion terms. Because of nonlinearity the higher order behaviour depends on the leading behaviour.) Basically, I use conformal invariance to argue that φ has to be analytic in $\boldsymbol{r}/r^2$ at infinity, and hence to greatly constrain its form there. The $\boldsymbol{K}$s have further to satisfy algebraic relations insuring that expression (2.22) satisfies the vacuum field equation. For $n = 1$ there are no extra relations on $\boldsymbol{K}$. For $n = 2$ the algebraic relation in the linear ($D = 2$) case is the usual tracelessness requirement $\sum_i K_{ii} = 0$. For $D > 2$ the condition on the matrix $\mathcal{K}$, the elements of which are K_{ij}, is $(\boldsymbol{r}\mathcal{K}\boldsymbol{r})^{(D-4)/2}\boldsymbol{r}[(D-2)\mathcal{K}^3 + \mathrm{Trace}(\mathcal{K})\mathcal{K}^2]\boldsymbol{r} = 0$ for all vectors $\boldsymbol{r}$. This, for symmetric $\mathcal{K}$, can be shown to imply $\mathcal{K} = 0$, so there is no dominant $n = 2$ behaviour for $D > 2$. Now consider a general theory with $\beta_0 = D-2$. Let φ be the solution for a confined charge distribution with vanishing total charge. Define

$$\rho^* \equiv (\alpha_D G)^{-1}\vec{\nabla}\cdot[\bar{\mu}(|\vec{\nabla}\varphi|/a_0)\vec{\nabla}\varphi] \tag{2.23}$$

where $\bar{\mu}$ is the exact $D-2$ power. Because asymptotically (where $\vec{\nabla}\varphi \to 0$) we have $\mu \to \bar{\mu}$, we get from Gauss's theorem that ρ^* also has a vanishing total charge. And, if μ approaches its power-law behaviour fast enough, ρ^* will be well bounded. But from equation (2.23) φ is a solution for ρ^* in the theory with the exact power-law $\bar{\mu}$ and so, from the arguments above, must also have the asymptotic form (2.21), (2.22).

Outside a spherical distribution of zero total charge the field vanishes for all theories.

Also of interest is the boundary condition $\vec{\nabla}\varphi \to -\boldsymbol{g}_0$ for $\boldsymbol{r} \to \infty$ ($\boldsymbol{g}_0$ is a constant vector). It pertains, for example, to a system of charges in an external electric field, to magnetized or superconducting bodies in an external magnetic field, or to obstacles and sources in an asymptotically uniform flow. For the asymptotic field use equation (2.13) to linearize in $\vec{\nabla}\zeta \equiv \vec{\nabla}\varphi + \boldsymbol{g}_0$

$$\vec{\nabla} \cdot [\vec{\nabla}\zeta + \hat{\mu}\boldsymbol{e}_0(\boldsymbol{e}_0 \cdot \vec{\nabla}\zeta)] = 0 \tag{2.24}$$

and solve with $\vec{\nabla}\zeta \to 0$ at infinity ($\boldsymbol{e}_0 \equiv \boldsymbol{g}_0/|\boldsymbol{g}_0|$). Taking $\boldsymbol{e}_0$ to be the x_1-direction we see that, asymptotically, ζ satisfies the Laplace equation in the coordinates $\boldsymbol{r}' \equiv \left[(1+\hat{\mu})^{-1/2}x_1, x_2, \ldots, x_D\right]$. Thus, ζ has the standard multipolar asymptotic expansion in $\boldsymbol{r}'$. When the total charge, Q, is finite the dominant behaviour is $\zeta \propto Q/r'^{D-2}$ ($D > 2$).

3. Forces on bodies

A body is an isolated region, v, where the field is externally disturbed in one way or another. For example, a body may be defined by some rigid distribution of charges in v, or by dictating the potential or its gradient on the boundary of v. The body may also be defined as a 'μ inclusion': dictating in v a $\mu(w)$ that is different from the ambient one. Examples of bodies defined by boundary conditions are: a rigid body in the flow problem and a superconducting body in a magnetic field, for both of which the normal component of $\vec{\nabla}\varphi$ vanishes at the surface; a conducting (equipotential) surface in an electric field; and a restricting boundary in the volume-minimization problem, for which the potential is dictated. Examples of bodies defined by μ inclusions are a dielectric inclusion, or, in the flow problem, a region of space where a fluid is subject to an external potential $\psi(\boldsymbol{r})$ that couples to fluid density. There is then an extra term, $\psi(\boldsymbol{r})$, in the Bernoulli equation, and $\varrho(u^2)$ is then a function of $\boldsymbol{r}$ in v through ψ.

Consider then a background having some ambient, position-independent μ. Within this background are embedded disjoint bodies of different types as defined above. Once one gets the solution, φ, for the system one can replace the bodies by effective charge distributions that give the same potential field outside the bodies. The effective charge density is simply

$$\rho^*(\boldsymbol{r}) \equiv (\alpha_D G)^{-1}\vec{\nabla} \cdot [\mu(|\vec{\nabla}\varphi|/a_0)\vec{\nabla}\varphi] \tag{3.1}$$

with the ambient μ used everywhere. If a body is defined by boundary conditions we continue the solution inside the body by solving the field equation with the ambient μ and imposing the same boundary conditions for the internal solution. Obviously, the effective charges given in this way appear as an extra charge only on the bodies because, by the field equation, the right-hand side of expression (3.1) gives the true charges outside the bodies. For example, a body that is defined by dictating φ on its surface is replaced by a ρ^* that constitutes a surface charge; a body defined by dictating $\mu\partial_n\varphi$ on the surface is replaced by a surface dipole layer (whose total charge obviously vanishes). A body defined by a μ inclusion is replaced by an effective charge distribution whose total effective charge is the same as the actual charge on the body. This is because Gauss's theorem, applied to equation (3.1), gives the total effective charge as a surface integral over $\mu\vec{\nabla}\varphi$ on a surface surrounding the body, but, from the field equation, this also equals the actual total charge of the body. So the total effective charge of a μ inclusion always vanishes if there are no actual sources within it.

The force on a body can be defined in several equivalent ways. For example, it may be taken as the gradient of the total energy under translation of the body. First replace all bodies in the system with their effective charges. Then translate the body in question, rigidly and infinitesimally, by $\delta\boldsymbol{a}$, keeping all the other charges (including effective ones) fixed. For the energy increment $\delta E = -\delta\boldsymbol{a}\cdot\boldsymbol{F}_v$, we identify $\boldsymbol{F}_v$ as the force on the body. Under the translation, the effective charge distribution, ρ, changes by $\delta\rho^* = -\delta\boldsymbol{a}\cdot\vec{\nabla}\rho^*$ in v, and $\delta\rho^* = 0$ outside. Since φ is an extremum of S, the change in E $(= -S)$ can be calculated as if only ρ^* has changed ($\delta\varphi = 0$ at infinity). Thus

$$\delta\boldsymbol{a}\cdot\boldsymbol{F}_v = -\delta E = -\int_v \varphi\delta\rho^*\,\mathrm{d}^D r = \delta\boldsymbol{a}\cdot\int_v \vec{\nabla}\rho^*\varphi\,\mathrm{d}^D r = -\delta\boldsymbol{a}\cdot\int_v \rho^*\vec{\nabla}\varphi\,\mathrm{d}^D r \tag{3.2}$$

where the last equality is obtained by integrating by parts. Thus

$$\boldsymbol{F}_v = -\int_v \rho^*\vec{\nabla}\varphi\,\mathrm{d}^D r = \int_v \varphi\vec{\nabla}\rho^*\,\mathrm{d}^D r. \tag{3.3}$$

The first equality may also serve as a definition of the force: it says that the force is the sum of forces on the elements of the body, each given by $-\rho^*\vec{\nabla}\varphi\,\mathrm{d}^D r$. But, unlike the linear case, here φ cannot be taken as that due to the rest of the system (excluding the body from the sources). In the linear case, but not here, the potential can be written as the sum of the contribution of the body and of the rest of the system. Since a body does not exert a force on itself (this is also true in the nonlinear case, see below) the contribution of the self-field drops from the expression for the force.

A third way, perhaps the most useful, of writing the force uses equation (2.10) ($\rho^*\vec{\nabla}\varphi = \vec{\nabla}\cdot\overleftrightarrow{\mathcal{P}}$) and expression (2.9) for $\overleftrightarrow{\mathcal{P}}$ to get

$$\boldsymbol{F}_v = -\int_\sigma \overleftrightarrow{\mathcal{P}}\cdot\mathrm{d}\boldsymbol{s} = \frac{a_0^2}{2\alpha_D G}\left[\int_\sigma \mathcal{F}\,\mathrm{d}\boldsymbol{s} - \int_\sigma 2\mathcal{F}\hat{\mathcal{F}}|\vec{\nabla}\varphi|^{-2}\vec{\nabla}\varphi\vec{\nabla}\varphi\cdot\mathrm{d}\boldsymbol{s}\right]. \tag{3.4}$$

The integration is done over any closed surface, σ, that surrounds the body and excludes all other sources and bodies (compare with the expression of the force as a surface integral in [3]). This expression has the advantage that it does not require the effective charge distribution of the body and employs only the field outside the body.

The surface integral in equation (3.4) vanishes automatically for the surface at infinity in theories with $\beta_0 \leqslant D-2$, so that the total force on a whole system vanishes, as expected from translational invariance. In other theories (e.g. in one dimension—see below) the boundary conditions for an isolated body must ensure this.

For nonlinear dielectrics, diamagnetics, etc the above definition of the force coincides with the usual expression for the electromagnetic force. In the flow problem $\boldsymbol{F}_v$ is the mechanical force acting on a region containing sources (ρ replaced by s) because the integral in expression (3.3) ($\int s\boldsymbol{u}$) is the rate at which the sources in the volume impart momentum to the flow. For a rigid obstacle standing in the flow, expression (3.4) gives the mechanical force the flow exerts on the obstacle because the second integral vanishes as $\vec{\nabla}\varphi\cdot\mathrm{d}\boldsymbol{s}$ vanishes on the surface (the flow is parallel to the surface) and we are left with $\boldsymbol{F}_v = -\int p\,\mathrm{d}\boldsymbol{s}$. The same is true for a type I superconducting body in a nonlinear magnetized medium: due to the Meissner effect $\vec{\nabla}\varphi\cdot\mathrm{d}\boldsymbol{s} = 0$ on the surface of the body.

In the volume-minimization problem, $\boldsymbol{F}$ is the actual lateral force (in the $x_1,\ldots,x_D$ plane) on the volume in question due to uneven tension.

In the case of non-equilibrium, stationary transport problems, such as nonlinear diffusion, the action functional is not an energy, and $\boldsymbol{F}$ is not a force, but measures the gradient in the entropy-generation rate when sources are translated.

3.1. A body in an external field—extension of the d'Alembert paradox

We saw that when $\vec{\nabla}\varphi$ vanishes at infinity the total force on any bound system vanishes. When we have $\vec{\nabla}\varphi \to -\boldsymbol{g}_0$ at infinity, describing a constant external field in which the system is immersed, it can be shown that even in the nonlinear case the force on the whole system is $Q\boldsymbol{g}_0$, where Q is the total charge. This can be seen by using the asymptotic form of the field given below equation (2.24) in expression (3.4) for $\boldsymbol{F}$ on a surface going to infinity. In the linear case this result is trivial since the total force is the sum of the total mutual forces of the charges, which vanishes, and the sum of the external forces $\int \rho\boldsymbol{g}_0\,\mathrm{d}^D r = Q\boldsymbol{g}_0$. It follows from this, for example, that the force on an arbitrary μ inclusion in a constant external field vanishes since, as we saw above, its total effective charge vanishes. The application of this result to the compressible flow problem constitutes a generalization of the well-known d'Alembert paradox in fluid mechanics to the effect that a single, static obstacle in a non-viscous, incompressible, potential flow that is uniform at infinity is subject to no force. We now see, more generally, that the paradox applies as well to the nonlinear (compressible but subsonic) case and to any single body made of fluid sources (or sinks), rigid obstacles, regions where body forces apply or any other configuration that can be replaced by an effective charge distribution with vanishing total charge.

3.2. Point bodies

When a body is very small compared with the typical scale of the system we can approximate it by a point charge (PC) q at position $\boldsymbol{r}_0$, with density $\rho(\boldsymbol{r}) = q\delta^D(\boldsymbol{r}-\boldsymbol{r}_0)$. We may view the body as the limit of some finite-size charge

$$\rho(\boldsymbol{r}) = q\lim_{\lambda\to 0}\lambda^{-D}\hat{\rho}(\boldsymbol{r}/\lambda) \tag{3.5}$$

where $\hat{\rho}$ is some smooth, finite charge distribution normalized to $\int\hat{\rho}(\boldsymbol{R})\,\mathrm{d}^D R = 1$. Not all problems admit PCs. Applying Gauss's theorem to equation (1.1) we see that $w\mu(w)$ diverges as $r^{-(D-1)}$ near a PC, so that such charges are not admitted in theories with $w\mu(w)$ bound from above. This is the case for the volume-minimization problem, where $w\mu(w) < 1$. It is also the case in the flow problem with $c^2 > 0$, where subsonicity, and hence ellipticity, are lost near a point source. For 'flow' problems with $c^2 < 0$, ellipticity is maintained at all values of $\vec{\nabla}\varphi$, and point sources are not objectionable.

The concept of a PC is useful only if the field everywhere in a system containing a PC is independent of the particular choice of the structure function $\hat{\rho}$ of the PC, so that it enters only through its total charge. I was not able to prove that this always holds. One possible route for proving this is to show that an infinitesimal change $\delta\hat{\rho}$ in the structure function, which does not change the total charge, $\int\delta\hat{\rho}(\boldsymbol{r})\,\mathrm{d}^D r = 0$, produces an everywhere-vanishing increment in the potential. The potential increment ζ is a solution of the linear equation (2.13) (where the background field depends on λ) with $\delta\rho = \lambda^{-D}\delta\hat{\rho}(\boldsymbol{r}/\lambda)$, in the limit $\lambda\to 0$. In this limit all the moments of $\delta\rho$ vanish, which might tell us that ζ vanishes: start with the charge distribution $\rho_{\lambda\lambda'} = \lambda^{-D}\hat{\rho}(\boldsymbol{r}/\lambda) + \lambda'^{-D}\delta\hat{\rho}(\boldsymbol{r}/\lambda')$ in place of the point PC. Let φ_λ be the solution of the problem for $\delta\rho = 0$, and $\varphi_\lambda + \zeta_{\lambda\lambda'}$ the solution for $\rho_{\lambda\lambda'}$. In the limit $\lambda\to 0$, φ_λ goes to the solution for the PC having $\hat{\rho}$ as a structure function. In the limit $\lambda = \lambda'\to 0$, $\varphi_\lambda + \zeta_{\lambda\lambda'}$ goes to the required solution for a PC with structure function $\hat{\rho}+\delta\hat{\rho}$. We want to show that in this last limit $\zeta_{\lambda\lambda'}\to 0$. Instead of going to the limit $\lambda = \lambda' = 0$ along the $\lambda = \lambda'$ line we first take the limit $\lambda'\to 0$ for finite λ and then take the limit $\lambda\to 0$. $\zeta_{\lambda\lambda'}$ solves equation (2.13) with the left-hand side depending on λ, and the right-hand side source being $\lambda'^{-D}\delta\hat{\rho}(\boldsymbol{r}/\lambda')$. As all the moments of this source vanish in the limit $\lambda'\to 0$, I conclude that $\zeta_{\lambda\lambda'}\to 0$ in the limit. Now taking the limit $\lambda\to 0$ we are left with $\zeta = 0$ in the limit. The remaining loophole concerns the equality of the two limits.

Be this as it may, the results concerning PCs are only valid when the field does not depend on the structure function. Employing the expression for the force as a surface integral (equation (3.4)) we see that the force on a PC is then also independent of the choice of $\hat{\rho}$.

Zero-size bodies with higher multipoles can, of course, also be constructed by considering a multiple limit with point charges of infinite charges at zero distances. Such bodies with pure multipole charge distribution do not have the special role they have in the linear case. They do not, in general, produce unique-multipole fields, and, unlike the pure-charge case, the force on them depends on the details of their structure, not only on the components of the multipole.

3.3. Test bodies

Consider a subsystem of charge distribution ρ_B. In the limit $\rho_B \to 0$ it can be considered as a test body: its contribution to the full potential can be neglected in expression (3.3) for the force on itself, and we have $\boldsymbol{F} = -\int \rho_B \vec{\nabla}\hat{\varphi}$, where $\hat{\varphi}$ is the potential determined by the rest of the system. A point charge has an infinite density. However, it may still be considered a test charge (the force on which $\boldsymbol{F} = -q\vec{\nabla}\hat{\varphi}$) provided q is small enough that a surface can be drawn around it such that (1) the surface is small compared with the scale over which the field varies appreciably, (2) it is far enough from the charge that the latter's contribution on the surface is small compared with the field due to the rest of the system alone (which is then approximately constant on the surface).

3.4. Attraction or repulsion

I now show that it is still correct in the nonlinear case that two equal PCs attract and two opposite charges repel each other (for $G > 0$; and vice versa for $G < 0$). Take, more generally, two finite, disjoint bodies with charge distributions that are the mirror images of each other about some $(D-1)$-dimensional hyperplane. The force on each can be calculated by taking the symmetry hyperplane as the integration surface in equation (3.4) (the integration on the hemisphere at infinity vanishes with our choice $\mathcal{F}(0) = 0$). Since φ is symmetric about the hyperplane, $\vec{\nabla}\varphi$ is in the plane. We thus get from equation (3.4)

$$\boldsymbol{F}_v = \frac{a_0^2}{2\alpha_D G}\int_\sigma \mathcal{F}\,\mathrm{d}s \tag{3.6}$$

which is attractive for $G > 0$. If one body is the negative-charge reflection of the other, φ is antisymmetric, and $\vec{\nabla}\varphi$ is perpendicular to the hyperplane so that $\overleftrightarrow{\mathcal{P}}\cdot \mathrm{d}s \propto \mathcal{F}(1-2\hat{\mathcal{F}})\,\mathrm{d}s$, and from equation (3.4)

$$\boldsymbol{F}_v = \frac{a_0^2}{2\alpha_D G}\int_\sigma \mathcal{F}(1-2\hat{\mathcal{F}})\,\mathrm{d}s \tag{3.7}$$

which is repulsive for $G > 0$ since $\hat{\mathcal{F}} > 1/2$ when $\vec{\nabla}\varphi \neq 0$.

Consider, more generally, two disjoint bodies B_i defined by charge distributions $\rho_i > 0$ in the non-overlapping volumes v_i, $i = 1, 2$. It is meaningful to ask whether they attract or repel each other only if they can be separated by some $(D-1)$-dimensional hyperplane. We have attraction if the force on B_1 crosses any such separating hyperplane from the side of B_1 to that of B_2 (see figure 1). I conjecture without a general proof that indeed this is always the case, and that if the two bodies are oppositely charged, i.e. if, say, $\rho_2 < 0$, they always repel each other. (The sign of the charge within each body must be uniform.) It follows from the result proved in appendix C that the conjecture holds when one of the bodies, say B_1, is spherically symmetric with a density profile decreasing from the centre out (I thank Shoshana Kamin for discussions leading to this proof). In fact, in this case the statement is stronger: for

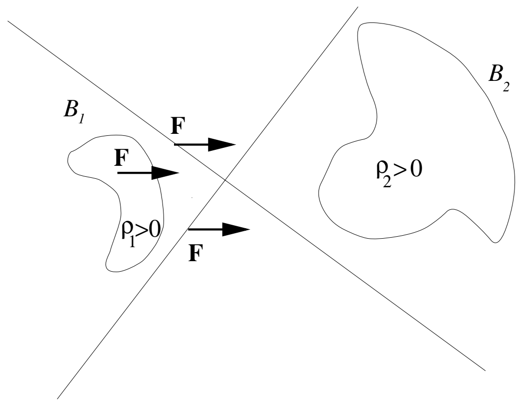

Figure 1. The force $\boldsymbol{F}$ on body 1 crosses every separating plane from the side of 1 to the side of 2.

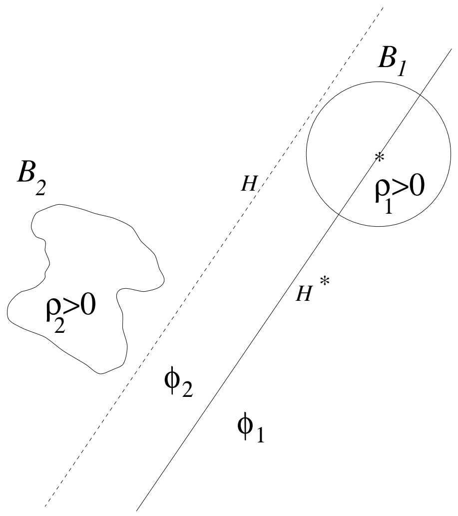

Figure 2. The setup for demonstrating attraction of like charges when one of the bodies is spherical with a decreasing density profile.

any hyperplane H^* through the centre of B_1 with B_2 wholly to its one side (side 2), the force on B_1 points from side 1 to side 2 (see figure 2).

We learn from the above result that the force on a spherical body with a decreasing density profile is always within the convex closure of the cone defined by its centre and the other body (the envelope of all the planes through the point that are tangent to the body).

Because a PC may be considered as the limit of such a spherical body, we deduce that the force on a PC in the presence of an extended body of uniform-sign charge is always within the cone from the point to the convex closure of the body. In particular, two PCs of the same sign always attract each other.

All the above generalizes, *mutatis mutandis*, to oppositely charged bodies.

The two-body case may be generalized to a push–pull conjecture concerning three bodies (as in figure 3) with $\rho_1, \rho_2 > 0, \rho_3 < 0$: for every two parallel hyperplanes separating the three bodies the force $\boldsymbol{F}$ acting on B_1 crosses the planes into the side of B_1. This seems to

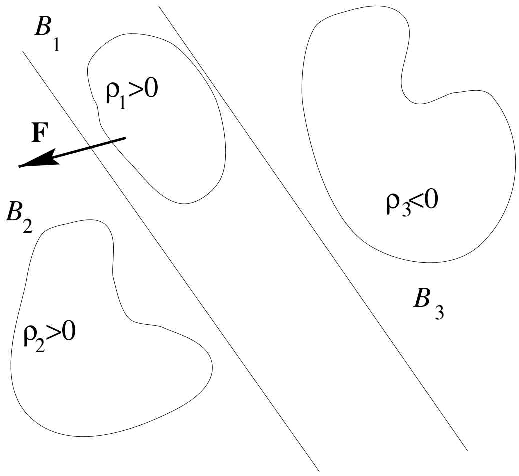

Figure 3. A configuration depicting the push–pull conjecture: for any pair of parallel separating planes the force $\boldsymbol{F}$ on B_1 crosses the planes into side 2.

be a good way to summarize the concept of attraction–repulsion of bodies with constant-sign charges as all else regarding the question follows from it.

I have not been able to prove this push–pull conjecture in all generality. It is proved in appendix C when body 1 is spherically symmetric with radially decreasing density; so, in particular, the conjecture holds when B_1 is a point charge. The conjecture is also proved in the one-dimensional case (see below). It also holds, generally, when body 1 is a collection of test charges. It then follows from the comparison principle discussed in appendix B. In this case the force on body 1 is $\boldsymbol{F} = -\int_1 \rho\vec{\nabla}\varphi_{12}$, where φ_{12} is the potential produced by bodies 2 and 3 alone. The comparison principle tells us that on any plane separating these bodies, $-\vec{\nabla}\varphi_{12}$ crosses from the side of 3 to that of 2 so $\boldsymbol{F}_1$ does so as well. Similarly, the conjecture holds when bodies 2 and 3 are made of test particles. The force on body 1 can then be written as $\boldsymbol{F} = \int_{12} \rho\vec{\nabla}\varphi_1$, where the integration is performed over bodies 1 and 2, and φ_1 is produced by body 1 alone. By the corollary of the comparison principle, $\vec{\nabla}\varphi_1$ crosses from the side of 3 to that of 2 in the volume of body 2, and the opposite in the volume of body 3. Thus, the integrand, and hence $\boldsymbol{F}$, crosses from 3 to 2.

Take a system containing charges of a uniform sign, say $\rho > 0$, with Σ a convex surface surrounding all the charges. It follows that on Σ, $\vec{\nabla}\varphi$ points outwards (inwards when $\rho < 0$). So, for example, in theories where $\mu \leqslant 1$ we have for the total charge, Q, the inequality $Q = (\alpha_D G)^{-1} \int_\Sigma \mu\vec{\nabla}\varphi \cdot \mathrm{d}s \leqslant (\alpha_D G)^{-1} \int_\Sigma \vec{\nabla}\varphi \cdot \mathrm{d}s \equiv Q^*$ where Q^* is the charge that would be deduced from the field $\vec{\nabla}\varphi$ in a linear theory.

3.5. Forces in the one-dimensional case

The one-dimensional case can be solved in closed form once the boundary conditions are fixed. If we require that $\vec{\nabla}\varphi(\infty) = -\vec{\nabla}\varphi(-\infty)$ so as to nullify the force on an isolated system, then $\vec{\nabla}\varphi \to$ const at ∞. The force, $\boldsymbol{F}$, on an arbitrarily charged body of total charge q, in the presence of a charge distribution that does not overlap with it, depends only on q and on the difference, Q, between the total charges to the right and to the left of the body.

$$F = F(q, Q) = \frac{a_0^2}{8G}[\chi(w^+) - \chi(w^-)] \tag{3.8}$$

where $\chi(w)$ is defined in equation (2.4), and

$$w^{\pm} = s(G)\nu^{-1}\left(\frac{|G|}{2a_0}|q \pm Q|\right) \tag{3.9}$$

with $\nu(w) \equiv w\mu(w)$.

We see that F is invariant under translations of the body, as long as it does not cross other charges, and that

$$F(-q, Q) = -F(q, Q) = F(q, -Q) \qquad F(Q, q) = F(q, Q). \tag{3.10}$$

These are peculiarities of the one-dimensional case. In higher dimensions the magnitude of the force on a charge does not, in general, remain invariant when the sign of the charge is reversed (all others kept intact).

Since $\chi' > 0$ for $w > 0$, χ increases everywhere. Thus, F does not vanish unless $w^+ = w^-$, i.e. unless $q = 0$, or $Q = 0$. The push–pull conjecture holds here as is easily seen.

4. Forces on point charges

4.1. The limit of a very large point charge

Consider a charge distribution ρ and a PC q that does not overlap with it. Assume also that q is much larger than any partial charge that makes up ρ (i.e. $\int_v \rho\, \mathrm{d}^D r \ll q$ for any volume v). We can then consider ρ to be a collection of test charges relative to q. From momentum conservation, the force $\boldsymbol{F}_q(\boldsymbol{R})$ acting on q at position $\boldsymbol{R}$ is opposite the force $\boldsymbol{F}_\rho$ acting on the distribution ρ. This latter is given in the test-particles limit by $\boldsymbol{F}_\rho = -\int \mathrm{d}^D r \rho(\boldsymbol{r})\vec{\nabla}\varphi_q$, where φ_q is the field produced by q alone, which is obtained in a straightforward manner through Gauss's theorem to give

$$\boldsymbol{F}_q(\boldsymbol{R}) = s(qG)a_0 \int \mathrm{d}^D r \rho(\boldsymbol{r})\nu^{-1}\left(\frac{|qG|}{a_0|\boldsymbol{r} - \boldsymbol{R}|^{D-1}}\right)\frac{\boldsymbol{r} - \boldsymbol{R}}{|\boldsymbol{r} - \boldsymbol{R}|} \tag{4.1}$$

with $\nu(w) \equiv w\mu(w)$. This force is derivable from an effective potential E_q

$$E_q(\boldsymbol{R}) = \int \mathrm{d}^D r \rho(\boldsymbol{r})\mathcal{G}_q(|\boldsymbol{r} - \boldsymbol{R}|) \tag{4.2}$$

where the effective Green's function, $\mathcal{G}_q$, satisfies

$$\vec{\nabla}_{\boldsymbol{r}}\mathcal{G}_q(\boldsymbol{r}) = s(qG)a_0\nu^{-1}(z)\boldsymbol{n} \tag{4.3}$$

with $z \equiv |qG|/a_0|\boldsymbol{r}|^{D-1}$, and $\boldsymbol{n} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$.

The effective potential is then a linear functional of the density ρ. It satisfies the Poisson equation

$$\Delta E_q(\boldsymbol{R}) = \int \mathrm{d}^D r \rho(\boldsymbol{r})\Delta\mathcal{G}_q(|\boldsymbol{r} - \boldsymbol{R}|) \tag{4.4}$$

with

$$\Delta\mathcal{G}_q(\boldsymbol{r}) = s(qG)(D-1)a_0|\boldsymbol{r}|^{-1}\frac{w\hat{\mu}(w)}{1 + \hat{\mu}(w)} \tag{4.5}$$

where $w = \nu^{-1}(z)$.

The above results also apply to any finite spherical body of very large total charge, and are extended in a straightforward manner to the case of an arbitrary body whose φ field is known.

4.2. The two-body force

For two PCs $|q_1| \leqslant |q_2|$, a distance ℓ apart, the vanishing of the total force and moment tells us that the forces on the two charges are opposite to each other, and lie along the connecting line; write its magnitude $f(q_1, q_2, \ell)$. We have seen above that f has the sign of Gq_1q_2 (f is positive for attraction). Because ℓ is the only length scale, we can reduce the number of variables to two independent, dimensionless variables constructed from q_1, q_2, ℓ, and a_0; for example, $-1 \leqslant \eta \equiv q_1/q_2 \leqslant 1$ and $z \equiv |Gq_2|/a_0\ell^{D-1}$. Since a_0q_1 has dimensions of force, we can write, for example

$$f = s(Gq_2)a_0q_1\hat{f}(\eta, z). \tag{4.6}$$

Our earlier discussion implies some constraints on $\hat{f}$. For instance, when one charge is much smaller than the other $|\eta| \ll 1$, the test-charge result tells us that

$$\hat{f}(|\eta| \ll 1, z) = \nu^{-1}(z) \tag{4.7}$$

to lowest order in η [$z = \nu(w) = w\mu(w)$].

For a power-law medium with $\mu(w) = w^\beta$ one deduces from scaling properties of the field equation that the ℓ-dependence of the force is $\ell^{-\gamma(D-1)}$, where $\gamma = 1/(1+\beta)$. So here

$$\hat{f} = \zeta(\eta)z^\gamma. \tag{4.8}$$

From equation (4.7) $\zeta(0) = 1$. Since, in general, $\zeta(-1) \neq \zeta(1)$, the forces for equal and for opposite charges are not of the same magnitude, in contradistinction with the linear case. For the special case $\beta = D-2$, the two-body force was found in closed form [8]

$$f(q_1, q_2, \ell) = s(G)\frac{1}{\ell}d^{-1}\left|Ga_0^\beta\right|^{d-1}\left(\left|q_1+q_2\right|^d - \left|q_1\right|^d - \left|q_2\right|^d\right) \tag{4.9}$$

[$d \equiv D/(D-1) = 1+\gamma$]. (The forces in a three-point-charge system of zero total charge were also derived in [8].) For two equal charges $q_1 = q_2 = q$, $f = 2s(G)\ell^{-1}d^{-1}|\hat{G}|^{d-1}|q|^d(2^{d-1}-1)$, whereas for opposite charges: $q_1 = -q_2 = q$, $f = -2s(G)\ell^{-1}d^{-1}|\hat{G}|^{d-1}|q|^d$. The two are equal in magnitude only in the (linear) two-dimensional case. Interestingly, in the limit of a large dimension, where $d \to 1$, the two-body force for two charges of the same sign vanishes as D^{-1} [from equation (4.9)], whereas the force for opposite charges does not. This can be generalized: for a given configuration of N point charges *of the same sign* the force on each becomes smaller as D^{-1} in the limit of large D (also letting the theory's power increase, $\beta = D-2$).

5. Some examples of applications

I next discuss a number of potential applications, some of which I alluded to earlier. I concentrate on two of the physical problems listed in the introduction: stationary (potential) flows of barotropic compressible fluids and media with field-dependent dielectric, or diamagnetic, constants.

Regarding the flow problem, we saw that the d'Alembert paradox can be extended to the compressible case. So, the force acting on a body of total effective charge in a flow of constant asymptotic speed vanishes. This implies the vanishing of the force on a body of rigid walls standing in such flow, on a system made of sinks and sources with vanishing total out-flux. If the fluid is electrically charged—weakly, so that the fluid does not self-interact—then the flow is modified by the presence of a region with an electric field. The effect of such a region can be described by an effective sink–source distribution whose total out-flux vanishes. So, the net force on such a region due to the fluid motion vanishes. We also learned from the discussion of the sign of the forces that in such flows two sinks, or two sources, always attract each other whereas a source and a sink repel each other.

For the nonlinear dielectric, the generalized d'Alembert paradox says that, as in the linear case, the force on an arbitrary charge distribution, e_i, of vanishing net charge in a constant external field, $\vec{E}$, vanishes, even though the force on an individual component of charge is not $e_i\vec{E}$. From this we learn that, for example, the force on a dielectric inclusion, or an equipotential body of zero net charge, vanishes in a constant external field. We also deduce that the force vanishes on a superconducting inclusion in a nonlinear magnetic medium in a constant magnetic field.

Acknowledgment

I thank Shoshana Kamin for helpful suggestions.

Appendix A. Integral relations

We employ the technique described in [10] to derive certain useful virial relations directly from the action by substituting in equation (2.15) for δS various choices of $\delta\varphi$. If either $\beta_0 < D-2$, or $\beta_0 = D-2$ and $Q = 0$, the potential vanishes at infinity. We can then obtain one relation by taking $\delta\varphi = \epsilon\varphi$, with ϵ infinitesimal. The vanishing of φ at infinity leads to the vanishing of δS. But δS can also be calculated directly to yield a virial relation

$$\int_V \rho\varphi\,\mathrm{d}^D r + \frac{1}{\alpha_D G}\int_V (\vec{\nabla}\varphi)^2\mu(|\vec{\nabla}\varphi|/a_0)\,\mathrm{d}^D r = 0. \tag{A.1}$$

(This can also be derived by multiplying φ by the expression for ρ from the field equation and integrating by parts.)

In the above relation (and below) ρ should be understood to include true sources as well as all the effective sources replacing boundary conditions on closed surfaces in the system.

Another useful relation, satisfied by solutions of the field equation, may be obtained by considering variations of S produced by dilations of space coordinates: $\varphi(\boldsymbol{r}) \to \varphi(\lambda\boldsymbol{r})$. For an infinitesimal dilation $\lambda = 1+\epsilon$, we have $\delta\varphi = \epsilon\boldsymbol{r}\cdot\vec{\nabla}\varphi$. Now, δS can also be calculated directly by substituting $\varphi(\boldsymbol{r}) \to \varphi(\lambda\boldsymbol{r})$ in S; then taking the derivative with respect to λ at $\lambda = 1$ to obtain for the finite volume

$$\left(\frac{\mathrm{d}S}{\mathrm{d}\lambda}\right)_{\lambda=1} = -\int_V \rho\boldsymbol{r}\cdot\vec{\nabla}\varphi\,\mathrm{d}^D r + \frac{a_0^2}{2\alpha_D G}\int_V \left[D\mathcal{F} - 2\frac{(\vec{\nabla}\varphi)^2}{a_0^2}\mu\right]\mathrm{d}^D r - \frac{a_0^2}{2\alpha_D G}\int_\Sigma \mathcal{F}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{s}. \tag{A.2}$$

So, comparing with expression (2.15) for the variation, we get an expression for the virial $\mathcal{V}$ in the volume V.

$$\mathcal{V} \equiv \int_V \rho\boldsymbol{r}\cdot\vec{\nabla}\varphi\,\mathrm{d}^D r = \frac{a_0^2}{2\alpha_D G}\int_V \mathcal{F}(D - 2\hat{\mathcal{F}})\,\mathrm{d}^D r + \int_\Sigma \boldsymbol{r}\cdot\overleftrightarrow{\mathcal{P}}\cdot\mathrm{d}\boldsymbol{s}. \tag{A.3}$$

This can also be obtained by substituting in the definition of $\mathcal{V}$ the expression for ρ from the field equation and integration by parts. The value of $\mathcal{V}$ is independent of the choice of origin due to momentum conservation.

Taking the surface to infinity, the surface term in equation (A.3) can be evaluated. At this point I restrict myself to $D \geqslant 2$; the one-dimensional case is exactly solvable and is treated below. For $\beta_0 < D-2$ the surface integral vanishes at infinity, and

$$\mathcal{V} = \frac{a_0^2}{2\alpha_D G}\int \mathcal{F}(D - 2\hat{\mathcal{F}})\,\mathrm{d}^D r. \tag{A.4}$$

For $\beta_0 = D-2$ the surface integral at infinity converges to yield

$$\mathcal{V} = \frac{a_0^2}{2\alpha_D G}\int \mathcal{F}(D-2\hat{\mathcal{F}})\,\mathrm{d}^D r + d^{-1}\hat{G}^{-1}|\hat{G}Q|^d \tag{A.5}$$

where $d \equiv D/(D-1)$, and $\hat{G} \equiv Ga_0^{\beta_0}$. The conformally invariant theories discussed in [8] have $\hat{\mathcal{F}} = D/2$ and so $\mathcal{V} = d^{-1}\hat{G}^{-1}|\hat{G}Q|^d$. This has been put to extensive use in [8].

Another integral of interest is

$$\vec{\mathcal{U}} \equiv \int_V \rho\left[\boldsymbol{r}(\boldsymbol{r}\cdot\vec{\nabla}\varphi) - \frac{1}{2}r^2\vec{\nabla}\varphi\right]\mathrm{d}^D r. \tag{A.6}$$

By substituting ρ from the field equation and integrating by parts we arrive at the expression

$$\vec{\mathcal{U}} = \frac{a_0^2}{2\alpha_D G}\int_V \mathcal{F}(D-2\hat{\mathcal{F}})\boldsymbol{r}\,\mathrm{d}^D r - \frac{a_0^2}{4\alpha_D G}\int_\Sigma \mathcal{F}r^2\{(1-2\boldsymbol{n}\otimes\boldsymbol{n}) + \hat{F}[4(\boldsymbol{n}\cdot\boldsymbol{e})\boldsymbol{n}\otimes\boldsymbol{e} - 2\boldsymbol{e}\otimes\boldsymbol{e}]\}\cdot\mathrm{d}\boldsymbol{s} \tag{A.7}$$

where $\boldsymbol{n} = \boldsymbol{r}/|\boldsymbol{r}|$ and $\boldsymbol{e} = \vec{\nabla}\varphi/|\vec{\nabla}\varphi|$.

Note that $\vec{U}$ depends, in general, on the choice of origin. Using the vanishing of the total force and total moment we see that if the origin is shifted by $-\boldsymbol{a}$, $\vec{U}$ is changed by $\mathcal{V}\boldsymbol{a}$.

Appendix B. The comparison principle and some consequences

First I show that if φ_1, φ_2 are continuous functions that solve our equation for densities $\rho_1 \geqslant \rho_2$ in a volume V of boundary Σ, and $\varphi_1 \leqslant \varphi_2$ on Σ, then $\varphi_1 \leqslant \varphi_2$ everywhere in V. This is known as a comparison principle for the solution of elliptic equations (e.g. [5]). I give here a proof that applies specifically to the form of our equation and is thus more elementary than the proofs found in the literature. Start with the identity

$$\int_v (\varphi_1-\varphi_2)(\rho_1-\rho_2)\,\mathrm{d}^D r \propto \int_v (\varphi_1-\varphi_2)\vec{\nabla}\cdot(\mu_1\vec{\nabla}\varphi_1 - \mu_2\vec{\nabla}\varphi_2)\,\mathrm{d}^D r \tag{B.1}$$

$$= \int_\sigma (\varphi_1-\varphi_2)(\mu_1\vec{\nabla}\varphi_1 - \mu_2\vec{\nabla}\varphi_2)\cdot\mathrm{d}\boldsymbol{s} - \int_v (\vec{\nabla}\varphi_1 - \vec{\nabla}\varphi_2)\cdot(\mu_1\vec{\nabla}\varphi_1 - \mu_2\vec{\nabla}\varphi_2)\,\mathrm{d}^D r \tag{B.2}$$

(where $\mu_i = \mu(|\vec{\nabla}\varphi_i|)$), obtained by using the expression for ρ_i from the field equation, Gauss's theorem and integration by parts. I want to show that the region v in V, in which $\varphi_1 > \varphi_2$, is empty. If there is even one point where $\varphi_1 > \varphi_2$, then, by continuity of $\varphi_1 - \varphi_2$, v must contain a whole (nonzero-measure) neighbourhood. Apply identity (B.2) to this volume v. On its boundary σ we have $\varphi_1 = \varphi_2$ (from continuity of φ_i), whether σ overlaps with Σ, or is completely interior to V. Thus the first term on the right-hand side of (B.2) vanishes. It can be shown (see [9]) that, in light of the ellipticity condition, the integrand in the second term is non-negative and vanishes only where $\vec{\nabla}\varphi_1 = \vec{\nabla}\varphi_2$. This is because when $w\mu(w)$ is non-decreasing $(\boldsymbol{a}-\boldsymbol{b})\cdot[\mu(|\boldsymbol{a}|)\boldsymbol{a} - \mu(|\boldsymbol{b}|)\boldsymbol{b}] \geqslant 0$ for any two vectors $\boldsymbol{a}, \boldsymbol{b}$, and vanishes only for $\boldsymbol{a} = \boldsymbol{b}$. However, by the assumptions, the left-hand side is non-negative and so the right-hand side must vanish, and hence $\vec{\nabla}\varphi_1 = \vec{\nabla}\varphi_2$ in v. Thus, $\varphi_1 - \varphi_2$ is constant in the whole region where it is positive. This, however, contradicts the assumption that $\varphi_1 \leqslant \varphi_2$ on Σ with the continuity of the potentials.

I now apply the theorem to the following useful configuration. Let $\varphi(x_1, \ldots, x_D)$ be the solution for a source distribution that satisfies $\rho(x_1, \ldots, x_D) \geqslant \rho(-x_1, \ldots, x_D)$. The boundary condition at infinity is $\varphi(\boldsymbol{r}) \to s(r)$. Observe the two halves of the solution in the two half-spaces separated by the $x_1 = 0$ hyperplane (H^*) as two solutions of the field equation in a half-space $x_1 \geqslant 0$; so, $\varphi_1(x_1, \ldots, x_D) \equiv \varphi(x_1, g \ldots, x_D)$, and $\varphi_2(x_1, \ldots, x_D) \equiv \varphi(-x_1, \ldots, x_D)$. We have $\varphi_2(0, x_2, \ldots, x_D) = \varphi_1(0, x_2, \ldots, x_D)$. So, φ_1 and φ_2 are solutions with the same boundary values (they also have the same boundary condition at ∞). The comparison principle then tells us that $\varphi_1(x_1, \ldots, x_D) \leqslant \varphi_2(x_1, \ldots, x_D)$, or $\varphi(x_1, \ldots, x_D) \leqslant \varphi(-x_1, \ldots, x_D)$. In particular, $\frac{\partial\varphi}{\partial x_1}(x_1 = 0) \leqslant 0$. H^* can be any plane separating a source distribution $\rho_1 \geqslant 0$ from $\rho_2 \leqslant 0$.

Let $\rho > 0$ in a volume v be the whole source distribution, and A its convex closure (A is the smallest convex volume containing all the points where $\rho \neq 0$). Then the gradient of the potential at any point $\boldsymbol{r}$ outside A points away from A (because it points away from the side of A on any plane separating $\boldsymbol{r}$ from A).

We can deduce from this that for a system of two PCs of the same sign the tangent to the field lines always crosses the line connecting the charges between the two. For oppositely charged points the tangent crosses outside this line segment. This generalizes the usual situation familiar from the linear case where it follows simply from the vector addition of the two forces due to the PCs.

Appendix C. Proof of the push–pull conjecture for a spherical body

Consider a system of three disjoint bodies: B_1, which is spherically symmetric with $\rho_1 > 0$ decreasing from the centre out; B_2 with $\rho_2 > 0$; and B_3 with $\rho_3 < 0$. H^* is a plane through the centre of B_1 that separates B_2 from B_3. I show that the force $\boldsymbol{F}$ on B_1 crosses H^* from side 3 to side 2. The push–pull conjecture for this special case is a weaker statement and follows as a corollary.

Choose the coordinates such that H^* is the $x_1 = 0$ plane, with B_2 on the $x_1 < 0$ side. We then have from the previous appendix that $\varphi(-x_1, \ldots, x_D) \leqslant \varphi(x_1, \ldots, x_D)$ for $x_1 > 0$. Using the second expression in equation (3.3) to calculate the x_1 component of the force on B_1, and employing the symmetry of the latter, we have

$$\boldsymbol{F}_1 = \int_{x_1>0} \mathrm{d}^D r[\varphi(x_1, \ldots, x_D) - \varphi(-x_1, \ldots, x_D)]\partial_{x_1}\rho \leqslant 0 \qquad \text{(C.1)}$$

where use was made of the fact that $\partial_{x_1}\rho \leqslant 0$ for $x_1 > 0$, as ρ monotonically decreases with radius.

References

[1] Gibbons G W and Rasheed D A 1995 *Nucl. Phys.* B **454** 185
[2] Gibbons G W 1998 *Nucl. Phys.* B **514** 603
[3] Bekenstein J and Milgrom M 1984 *Astrophys. J.* **286** 7
[4] Adler S L and Piran T 1984 *Rev. Mod. Phys.* **56** 1
[5] Gilbarg D and Trudinger N S 1977 *Elliptic Partial Differential Equations of Second Order* (Berlin: Springer)
[6] Earnshaw S 1842 *Trans. Camb. Phil. Soc.* **7** 97
[7] Milgrom M 1998 *Phys. Lett.* A **243** 33
[8] Milgrom M 1997 *Phys. Rev.* E **56** 1148
[9] Milgrom M 1986 *Astrophys. J.* **302** 617
[10] Milgrom M 1994 *Phys. Lett.* A **190** 17